Layered wheels

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ENS DE LYON

Introduction

Let G be a graph.

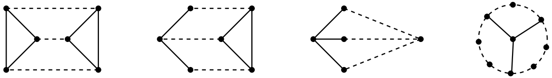
- H is an induced subgraph of G, if it can be obtained from G by deleting vertices
  ☞ we say that G contains H

- G is H-free if it does not contain any graph isomorphic to H

- G is $\mathcal{F}$-free if it is H-free, for every $H \in \mathcal{F}$

Truemper configurations

We are interested in the existence of the following graphs as induced subgraphs of the classes being studied.



prism pyramid theta wheel

[1] Chudnovsky, Robertson, Seymour, and Thomas (2002)
[2] Conforti, Cornuéjols, Kapoor, and Vušković (2002)
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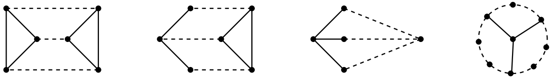

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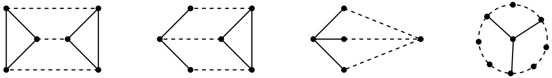
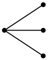


prism pyramid theta wheel

✲ any pair of the three paths must induce a **hole** (chordless cycle of length at least 4)

Some classes that exclude Truemper configurations:

- **Perfect graphs** → excluding pyramid, some wheels [1]
- **Even-hole-free graphs** → excluding prism, theta, some wheels [2]
  - even hole = a hole of even length
- **Claw-free graphs**  claw → excluding pyramid, theta, many wheels [3]
- many others...

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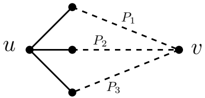
Even-hole-free (EHF) graphs and Theta-free (TF) graphs

Recall:

- an even hole is a hole of even length
- a theta is a graph induced by three paths s.t. any two of them induce a hole.



Remark. At least two of P_1, P_2, P_3 have same parity

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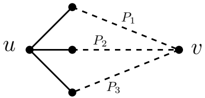

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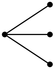
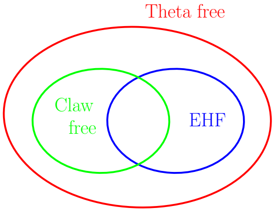
Observation. A theta always contains a claw and an even hole





EHF graphs vs TF graphs - Algorithmic/structural issues

- Coloring
  - NPC for TF graphs (because it is NPC for claw-free graphs [4])
  - open for EHF graphs
- Max independent set
  - open for both classes
  - **Remark.** it is polynomial for claw-free graphs [5]
- Decomposition theorem
  - several decomposition theorems are known for EHF graphs [6]
  - open for TF graphs

[4] Hoyler (1981)
[5] Minty (1980)
[6] Conforti, Cornuéjols, Kapoor, and Vušković (2002); da Silva and Vušković (2008)

Graph widths

☞ a parameter that measure how "complex" the structure of a graph is

Three notions of widths that we use:

- Treewidth *tw*
- Pathwidth *pw*
- Rankwidth *rw*

a well-known bound:

$$\text{For any graph } G,\ rw(G) \leq tw(G) \leq pw(G)$$

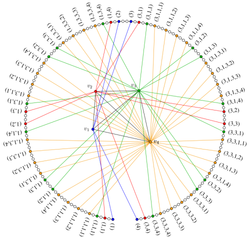
Remark.

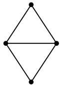
- Observation: EHF graphs have unbounded treewidth (clique, chordal graphs are EHF)
- It is not easy to produce a (non trivial) EHF graphs of large widths
- also not easy to find a (non trivial) subclass of small widths

Some results on the widths of EHF graphs

Negative results on EHF graphs[7]

- The rankwidth of EHF graph with no diamond is unbounded





diamond

☆ **Remark.** the graph contains large clique

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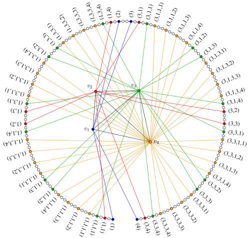

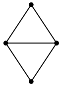

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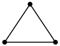
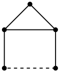
Positive results on EHF graphs [8]

- EHF graph G with no triangle: $tw(G) \leq 5$
- EHF graph G with no cap: $tw(G) \leq 48$
- EHF graph G with no pan: $tw(G) \leq 1.5\omega(G)$

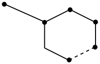
triangle





cap

= hole + short chord



pan

= hole + pendant edge

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Observation, motivation, and results

Recall: EHF graphs with no triangle have treewidth at most 5

- How about EHF graphs with no K_4?

  ☞ we show that the treewidth can be arbitrarily large

- This also answers the following question[9] :
  is the treewidth of EHF graph (in general) bounded by a function of its max clique size?

  ☞ no, our constructions have small clique size, but large treewidth

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- idem for triangle-free TF-graph.

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Such graphs is what we name as **Layered wheels**

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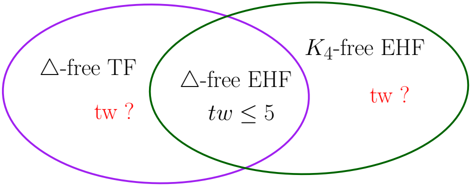
△–free TF graphs & K_4-free EHF graphs

What we have and what we study...



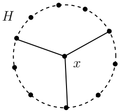
Observation:

- △-free TF graphs and K_4-free EHF graphs both are (prism, pyramid, theta)-free
- △-free TF graphs do not contain certain wheels,
  K_4-free EHF graphs do not contain some other wheels

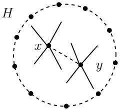
☞ Let's study the structure of **wheels** in our classes

Wheels in our classes

- wheel = hole H + a vertex x that has at least 3 neighbours in H
- 2-wheel = hole H + vertices x, y that each has at least 3 neighbours in H



wheel



2-wheel

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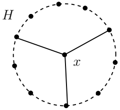
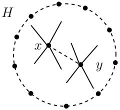


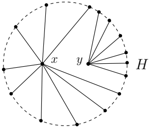
wheel



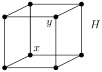
2-wheel

Structure of 2-wheels with x and y non-adjacent

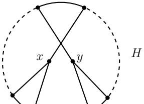
- In △-free EHF graph : 2-wheels are always nested
- In △-free TF graph : nested, except the cube
- In K_4-Free EHF graph : nested, with several exceptions

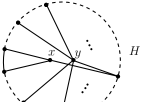


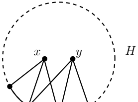
nested wheel



cube







Layered wheel

Notation: Layered wheel, of ℓ layers and girth *k*

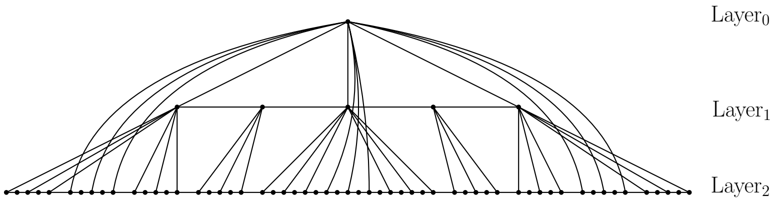
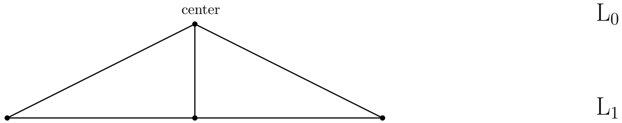


Figure: △-free TF layered wheel of 2 layers with girth 4

△-free TF layered wheel - construction

$G(\ell, k)$, with $\ell = 2$ and $k = 4$

△-free TF layered wheel - construction



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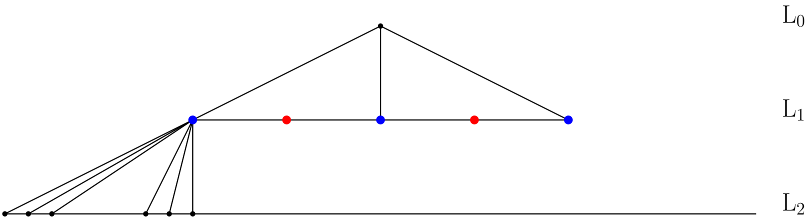
$\triangle$-free TF layered wheel - construction



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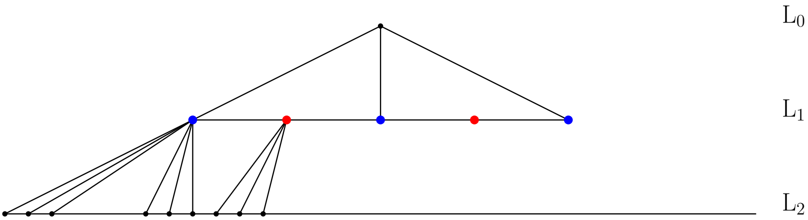

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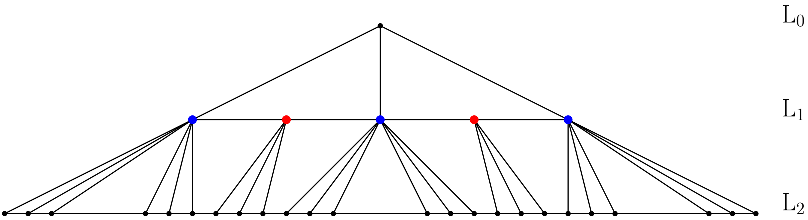

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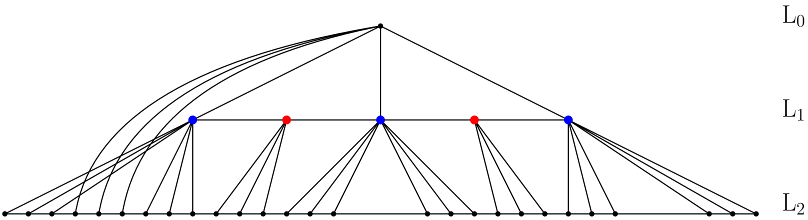

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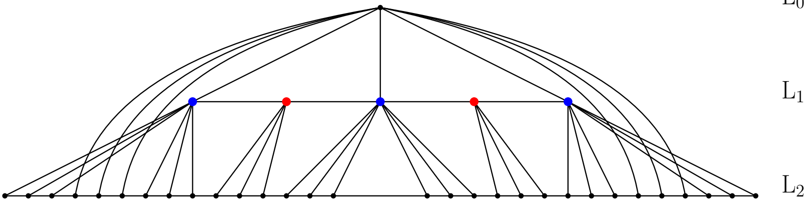

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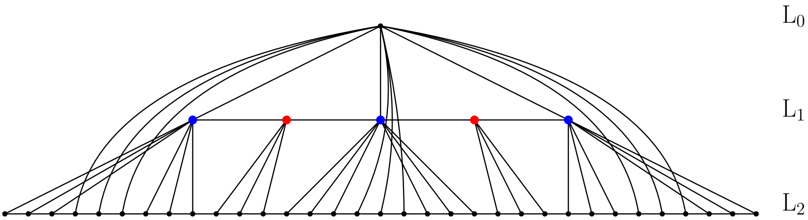

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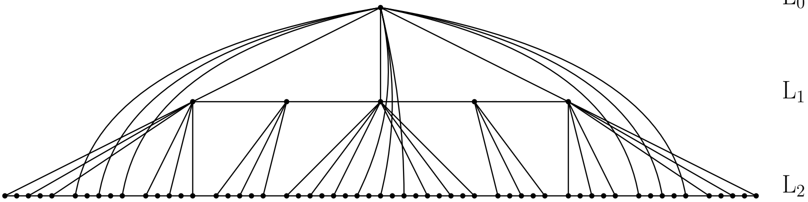

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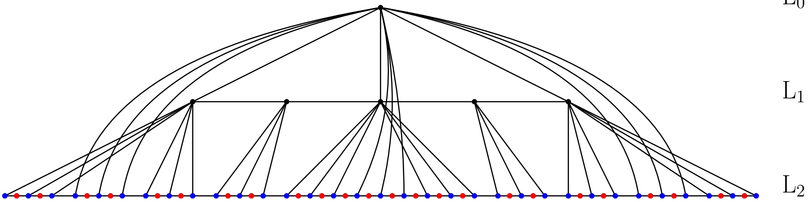

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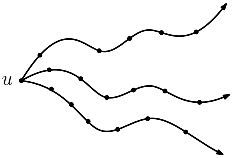
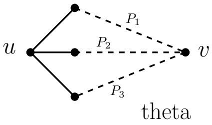
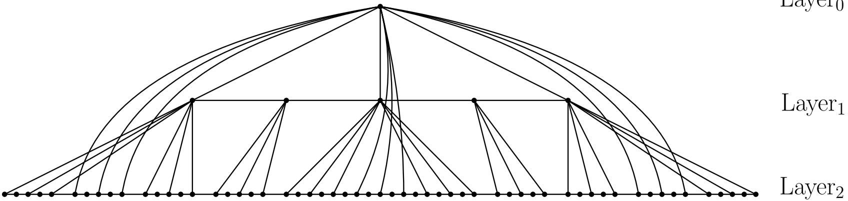
Layered wheel is theta-free

- Layered wheel is full of subdivided claw but... it contains no theta

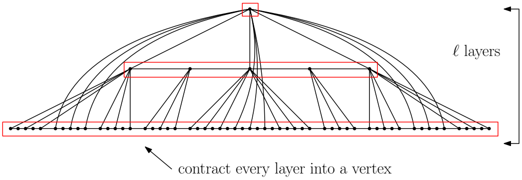






Treewidth of layered wheel

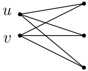
- the treewidth of layered wheel on ℓ layers is at least ℓ



- stronger result: the rankwidth is unbounded
  - **Classical theorem:** [10]

    Let t be integer, and $\mathcal{C}$ be a class of graph that do not contain $K_{t,t}$ as a subgraph. Then the class has bounded *tw* iff it has bounded *rw*.

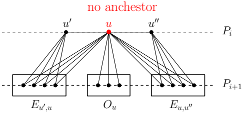
  - our $\triangle$-free TF layered wheels have no $K_{2,3}$

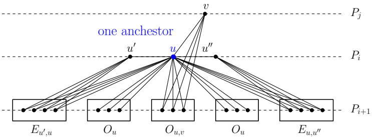


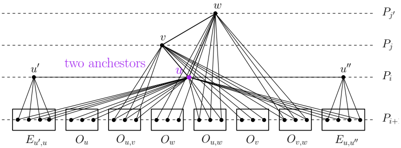
[10] Gurski and Wanke (2000)

EHF layered wheel - construction

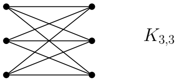
- The first two layers are similar as for △-free TF layered wheel
- The construction is more complicated. There are three types of vertices.







- The treewidth is at least ℓ
- It is even-hole-free
- It contains no $K_{2,2}$, so *rw* is unbounded



The treewidth is ”small” in some sense

Consider layered wheel on ℓ layers.

Remark. to reach treewidth ℓ, the layered wheel needs $\Omega(3^{\ell})$ vertices.

Theorem 2. [DS., Trotignon (2019+)]

The treewidth of layered wheel is in $O(\log(n))$ where n is the vertex size.

Proof.

1. $n \gg 3^{\ell}$
2. $tw(\text{layered wheel}) \leq pw(\text{layered wheel}) \leq 2\ell$

To sum up...

Theorem 1. For every integers $\ell \geq 1$ and $k \geq 3$ there exists a graph $G_{\ell,k}$ such that:

- it is theta-free and it has girth at least k (so, is $\triangle$-free when $k \geq 4$).
- $\ell \leq rw(G_{\ell,k}) \leq tw(G_{\ell,k}) \leq pw(G_{\ell,k}) \leq 2\ell \leq 2^{\ell} \leq |V(G_{\ell,k})|$.

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Theorem 2. For every integers $\ell \geq 1$ and $k \geq 3$ there exists a graph $G_{\ell,k}$ such that:

- it is K_4-free EHF and every hole in the graph has length at least k.
- $\ell \leq rw(G_{\ell,k}) \leq tw(G_{\ell,k}) \leq pw(G_{\ell,k}) \leq 2\ell \leq 2^{\ell} \leq |V(G_{\ell,k})|$.

Logarithmic treewidth conjecture

Conjectures. [DS., Trotignon (2019+)]

- There exists a constant c such that for any $\triangle$-free TF graph G, we have

$$tw(G) \leq c \log |V(G)|.$$

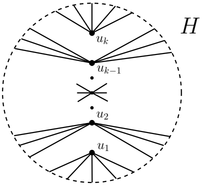
- Idem for K_4-free EHF graph.

☞ if these are true, many graph problems are poly-time solvable in both classes.

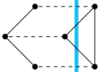
Excluding more structure

Theorem 3. [DS., Trotignon (2019+)]

- Let k be fixed. There exists a constant c such that any (theta, triangle, *k-span-wheel*)-free graph G has treewidth bounded by $O\left(k^6\right)$.
- idem for (K_4, *pyramid*, *k-span-wheel*)-free EHF graph $\rightarrow O\left(k^9\right)$



k-span-wheel

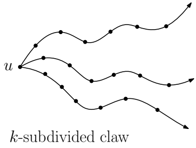


pyramid

Excluding more structure

Theorem 4. [DS., Trotignon (2019+)]

Let $k \geq 1$ fixed. There exists a constant c such that any (theta, triangle, *k-subdivided-claw*)-free graph has treewidth bounded by $O\left(k^{o(1)}\right)$.



Remark.

- The theorem is interesting regarding the max independent set problem. It is open for the class of k-subdivided-claw-free graphs and the class of theta-free graphs
- The theorem is no more true if one exclusion is forgotten.

Grid-minor-like conjecture

Grid minor theorem [11]

There exists a function f such that: if $tw(G) \geq f(k)$, then G contains a grid of treewidth at least k as a minor.

☞ Is there a similar theorem with induced subgraph instead of minor?

[11] Robertson and Seymour (1986)

Grid-minor-like conjecture

[12]Does there exist a function f such that: every graphs with $tw \geq f(k)$ contains as an induced subgraph some graph of $tw \geq k$ that is one of the following

- a big clique
- a big complete bipartite graph
- a big grid (possibly subdivided)
- a big wall (possibly subdivided)
- a big line graph of a subdivided wall

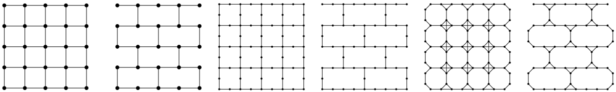


Figure: A grid, a wall, a subdivision of the former and its line graphs

[12]informally questioned by Zdeněk Dvořák

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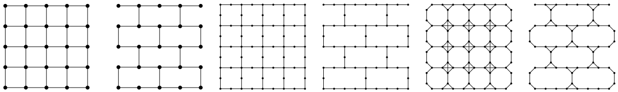

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☞ no, layered wheels is a counter-example

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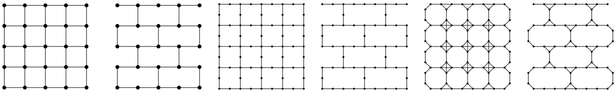

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☞ no, layered wheels is a counter-example

How if some items are added to the list?

- layered wheels or some variation of them?
- long paths (P_t with $t \geq 5$)?
- a vertex of high degree (at least 4)?
- graphs containing $\geq c^{f}(k)$ vertices?

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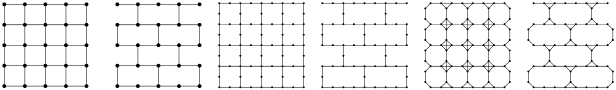

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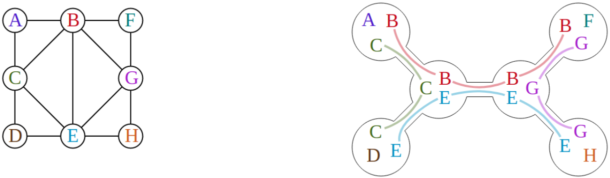
Treewidth and pathwidth

Treewidth

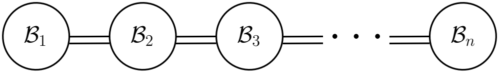


figures taken from https://commons.wikimedia.org/wiki/User:David_Eppstein/Gallery

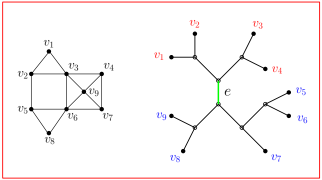
- The treewidth of G is a parameter measuring how far is G from a tree
- We create a tree decomposition whose nodes are ”bags”
- The *treewidth of G* is the size of the largest bag minus 1 in an optimal tree decomposition

pathwidth

- ”path”-version of tree decomposition



Rankwidth



$$\text{width}(e) \;=\; \text{rank} \begin{array}{c} \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array} \;=\; 3$$

example from Hlineny et. al. Width parameters beyond treewidth and their applications, The Computer Journal (51), 2008

- It is a parameter measuring the connectivity of G
- Rank decomposition is a cubic tree $\mathcal{T}$, with a bijection $V(G) \to \mathcal{L}(\mathcal{T})$
- the rankwidth of G is the cut-rank of the adjacency matrix of the separation in an optimal rank decomposition of G

Decomposition theorem

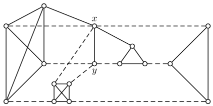
Decomposition theorem

Theorem 1. [Conforti, Cornuéjols, Kapoor, Vušković (2002); da Silva, Vušković (2008)]

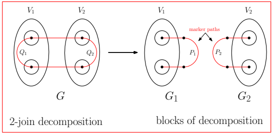
A connected ehf graph [a] is either basic or it has a 2-join or a star cutset.

[a] the statement is proved for more general class, namely 4-hole-free odd-signable graph

- **Basic**: a clique, or a hole, or an extended nontrivial basic graph



- **2-join**



- **star cutset**